Name (IN CAPITALS): **Version #1**

Instructor: Dora The Explorer

Math 10550 Exam 2
Oct. 17, 2024.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off (and Put Away) all cellphones, smartwatches and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 16 pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-16.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)
11.	(•)	(b)	(c)	(d)	(e)
12.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
13.	________
14.	________
15.	________
16.	________
Total	________

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2	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
13.	________
14.	________
15.	________
16.	________
Total	________

Multiple Choice

1.(7pts) Which of the following gives the equation to the tangent line to the ellipse given by the equation $x^2 + 2y^2 = 9$ at the point $(1, -2)$?

Solution: $y = \frac{x}{4} - \frac{9}{4}$.

Using implicit differentiation, we have

$$2x + 4yy' = 0.$$

Plug in $x = 1$ and $y = -2$, we have $2 - 8y' = 0$, hence, $y' = \frac{1}{4}$ and the tangent line is

$$y + 2 = 1/4(x - 1) \text{ equivalently, } y = 1/4x - 9/4.$$

(a) $y = \dfrac{1}{4}x - \dfrac{9}{4}$

(b) $y = \dfrac{1}{4}x + \dfrac{3}{4}$

(c) $y = -\dfrac{1}{4}x - \dfrac{7}{4}$

(d) $y = \dfrac{1}{2}x - 5$

(e) $y = \dfrac{1}{2}x + \dfrac{3}{2}$

2.(7pts) Find y', if

$$\sin(y) + xy^2 = 1.$$

Solution: $\frac{-y^2}{\cos(y)+2xy}$.

Using implicit differentiation, we have

$$\cos(y)y' + y^2 + 2xyy' = 0.$$

Hence, we have $y' = \frac{-y^2}{\cos(y)+2xy}$.

(a) $\dfrac{-y^2}{\cos(y) + 2xy}$

(b) $\dfrac{1}{\cos(y) + 2xy}$

(c) $\dfrac{2xy - y^2}{\cos(y)}$

(d) $\dfrac{-y^2 - \cos(y)}{2xy}$

(e) $\dfrac{y^2 + \cos(y)}{2xy}$

3.(7pts) The height of a ball thrown straight upwards on the planet Zing is given as

$$h(t) = -3t^2 + 6t + 24,$$

where height, h, is measured in feet above the surface, and time, t, is measured in seconds for $0 \leq t \leq t_i$, where t_i is the time at which the ball hits the surface. What is the speed of the ball at the moment of impact? (i.e. with what speed does the ball hit the surface?)

Solution: $18 ft./sec.$.
First, we factor h as $h(t) = -3(t-4)(t+2)$, so to ball hit the surface at time $t = 4$. Then we compute $h'(t) = -6t + 6$, plug in $t = 4$ yields $h'(4) = -18$.

(a) 18 ft./sec. (b) 6 ft./sec. (c) 1 ft./sec. (d) 24 ft./sec. (e) 27 ft./sec.

4.(7pts) An object moving along a straight line has position function at time t given by

$$s(t) = t + t^2 + \cos(\pi t)$$

Which of the following statements is correct?

Solution: The object is slowing down when $t = 1/2$.
We compute

$$s'(t) = 1 + 2t - \pi \sin(\pi t) \text{ and } s''(t) = 2 - \pi^2 \cos(\pi t).$$

At $t = 1/2$, we have $s'(1/2) = 2 - \pi < 0$ and $s''(1/2) = 2 > 0$. Since they have different sign, the object is slowing down.

(a) The object is slowing down when $t = 1/2$

(b) The object is at rest when $t = 1/2$

(c) The object is moving forward when $t = 1/2$

(d) The acceleration is equal to 0 when $t = 1/2$

(e) The velocity is decreasing when $t = 1/2$

5.(7pts) A particle is moving in a straight line along a horizontal axis with a position function given by

$$s(t) = 2t^3 - 3t^2 - 12t + 20,$$

where distance is measured in feet and time is measured in seconds. What is the distance travelled by the particle in the time period $0 \leq t \leq 3$ seconds?

Note: $s(3) = 11$.

Solution: 31 feet.

We compute

$$s'(t) = 6t^2 - 6t - 12 = 6(t-2)(t+1).$$

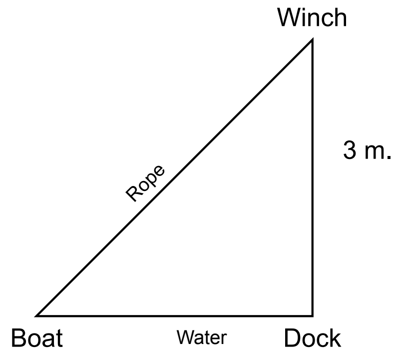
Hence, the particle moves backwards from $t = 0$ to $t = 2$ and it moves forwards from $t = 2$ to $t = 3$. Therefore, the total distance is

$$s(0) - s(2) + s(3) - s(2) = 20 - 0 + 11 - 0 = 31.$$

(a) 31 feet (b) 9 feet (c) 11 feet

(d) 3 feet (e) 17 feet

6.(7pts) A winch 3 meters above a dock is used to reel in a rope connected to a boat. The rope is being reeled in at 0.1 meters per second. The boat is moving towards the dock along the water. How fast is the boat moving along the water when the rope is 5 meters in length?



Solution: Let x denote the distance from the boat to the dock. Let r denote the lenght of the rope (the distance from the boat to the winch). We have a right triangle so we have the relation:

$$r^2 = x^2 + 3^2$$

Implicitly differentiate the above with respect to time to get:

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 0$$

which simplifies to

$$r\frac{dr}{dt} = x\frac{dx}{dt}$$

We want to find $\frac{dx}{dt}$ when $r = 5$ and $\frac{dr}{dt} = -0.1$ ($\frac{dr}{dt}$ is negative since the rope is decreasing in length). Note that when $r = 5$, we have that:

$$5^2 = x^2 + 3^2$$

which gives $x = 4$. Now we plug in $x = 4$, $r = 5$, and $\frac{dr}{dt} = -0.1$ into the equation $r\frac{dr}{dt} = x\frac{dx}{dt}$ to get:

$$5(-0.1) = 4\frac{dx}{dt}$$

which gives $\frac{dx}{dt} = -1/8$. Since the problem is asking for the speed of the boat moving along the water, we take the abolute value of $\frac{dx}{dt}$ to get $|\frac{dx}{dt}| = 1/8$. Thus, the answer is 1/8.

(a) $\frac{1}{8}$ m/s (b) $\frac{3}{40}$ m/s (c) $\frac{1}{2}$ m/s (d) $\frac{1}{10}$ m/s (e) $\frac{1}{25}$ m/s

7.(7pts) Find the linearization of $f(x) = \sin(2x)$ at $a = \dfrac{\pi}{2}$.

Solutions: the equation for the linearization of $f(x)$ at a is given by the following formula:

$$L_a(x) = f'(a)(x - a) + f(a)$$

In this problem, $a = \pi/2$. Thus, the above formula becomes:

$$L_{\pi/2}(x) = f'(\pi/2)(x - \pi/2) + f(\pi/2)$$

We compute that

$$f(\pi/2) = \sin(2(\pi/2)) = \sin(\pi) = 0$$

Taking the derivative, we get $f'(x) = 2\cos(2x)$. Thus,

$$f'(\pi/2) = 2\cos(2(\pi/2)) = 2\cos(\pi) = -2$$

Therefore, the linearization is:

$$L_{\pi/2}(x) = -2(x - \pi/2) + 0$$

$$= -2x + \pi$$

(a) $L(x) = -2x + \pi$

(b) $L(x) = x - \dfrac{\pi}{2}$

(c) $L(x) = -1$

(d) $L(x) = -2x + \dfrac{\pi}{2}$

(e) $L(x) = 2x$

8.(7pts) Find the linearization of $f(x) = x^{3/4}$ at $a = 16$ and use it to approximate $17^{3/4}$.

Solutions: We need to find:

$$L_a(x) = f'(a)(x-a) + f(a)$$

with $a = 16$. Note that $f(16) = 16^{3/4} = (2^4)^{3/4} = 2^3 = 8$. We also have $f'(x) = \frac{3}{4}x^{-1/4}$ and thus $f'(16) = \frac{3}{4}(16)^{-1/4} = \frac{3}{4}(\frac{1}{2}) = \frac{3}{8}$. Thus, we get:

$$L_{16}(x) = \frac{3}{8}(x-16) + 8$$

We are asked to approximate $17^{3/4} = f(17)$, so we get:

$$17^{3/4} = f(17) \approx L_{16}(17)$$

$$\approx \frac{3}{8}(17-16) + 8$$

$$\approx \frac{3}{8}(1) + 8$$

$$\approx \frac{3}{8} + \frac{64}{8}$$

$$\approx \frac{67}{8}$$

(a) $\frac{67}{8}$ (b) $\frac{19}{2}$ (c) $\frac{17}{2}$ (d) $\frac{61}{8}$ (e) $\frac{35}{4}$

9.(7pts) Find the critical numbers of the function

$$f(x) = \frac{x^2+8}{x-1}.$$

Solutions: The critical numbers are $x = 4, -2$.

We compute

$$f'(x) = \frac{(x-1)(2x)-(x^2+8)}{(x-1)^2} = \frac{x^2-2x-8}{(x-1)^2} = \frac{(x-4)(x+2)}{(x-1)^2}.$$

Note that $f'(x)$ is zero when $x = 4, -2$ and that $f'(x)$ is undefined when $x = 1$. However, $x = 1$ does not lie in the domain of the original function $f(x)$. Thus, the critical numbers are $x = 4, -2$.

(a) $x = 4,\ x = -2$

(b) $x = 4,\ x = 2$

(c) $x = \sqrt{8},\ x = -\sqrt{8}$

(d) $x = 4,\ x = 0$

(e) $x = 0,\ x = 2$

10.(7pts) Find the absolute maximum of the function

$$f(x) = x^3 - 6x^2 + 9x + 1$$

on the interval $[0, 2]$.

Note: We are asking for the y-value on the graph at the maximum here.

Solutions: 5.

First, we find the critical points of f:

$$f'(x) = 3x^2 - 12x + 9 = 3(x-3)(x-1).$$

Hence, f has critical points at $x = 1$ and $x = 3$. Next, we compute

$$f(0) = 1,\ f(1) = 5,\ f(2) = 3.$$

Hence, the maximum is equal to 5.

(a) 5 (b) 3 (c) 1 (d) 7 (e) 9

11.(7pts) A function $f(x)$ is continuous on the interval $0 \leq x < \infty$ and differentiable on the interval $(0, \infty)$. If $f(1) = 5$, and $0 \leq f'(x) \leq 3$ for $x > 0$, which of the following **must** be true ?
(only one must be true, the remaining ones **might** be false)

Solutions: $5 \leq f(3) \leq 11$.
By MVT, $0 \leq \frac{f(3)-f(1)}{3-1} = \frac{f(3)-5}{2} \leq 3$. Hence, $5 \leq f(3) \leq 11$.

(a) $5 \leq f(3) \leq 11$

(b) $5 \leq f(3) \leq 8$

(c) $f'(x) = 3$ for some x with $1 \leq x \leq 5$.

(d) $f'(5) = 3$

(e) $0 \leq f(3) \leq 6$

12.(7pts) Let

$$f(x) = 27x - x^3 + 2024.$$

Which of the following statements is true about $f(x)$?

Solutions: f has a local minimum at $x = -3$ and f has a local maximum at $x = 3$.
We compute

$$f'(x) = 27 - 3x^2 = 3(9 - x^2) = -3(x-3)(x+3).$$

Hence, $f' < 0$ for $x < -3$; $f > 0$ for $-3 < x < 3$; $f < 0$ for $x > 3$. Therefore, f has a local minimum at $x = -3$ and f has a local maximum at $x = 3$.

(a) There is a local minimum at $x = -3$ and a local maximum at $x = 3$

(b) There is a local maximum at $x = -3$ and a local minimum at $x = 3$

(c) There are local minima at $x = -3$ and $x = 3$, and no local maxima

(d) There are local maxima at $x = -3$ and $x = 3$, and no local minima

(e) There is a local minimum at $x = 3$ and no local maxima.

10. Initials: ____________

Partial Credit

For full credit on partial credit problems, make sure you justify your answers.

13.(10pts) The velocity of a particle moving along a straight line at time t is given by

$$v(t) = \frac{t^2 - t - 2}{t - 3} \quad \text{where} \quad 0 \le t \le 2.$$

(a) What is the acceleration function of the particle

Solution: Apply quotient rule:

$$\begin{aligned} a(t) = v'(t) &= \frac{(t-3)(2t-1) - (t^2 - t - 2)(1)}{(t-3)^2} \\ &= \frac{(2t^2 - 7t + 3) - (t^2 - t - 2)}{(t-3)^2} \\ &= \frac{t^2 - 6t + 5}{(t-3)^2} \end{aligned}$$

(b) On which time intervals (in the time period $0 \le t \le 2$) is the particle speeding up?

Solution: We set $v(t) = \frac{t^2-t-2}{t-3}$ equal to zero:

$$0 = \frac{t^2 - t - 2}{t - 3}$$

Since we are assuming $0 \le t \le 2$, we know that $t - 3 \ne 0$, so we get:

$$0 = t^2 - t - 2$$

the above solutions are $t = -1, 2$, but since we are only looking for values on the interval $[0, 2]$, we just need $t = 2$. Note that

$$v(t) > 0 \text{ when } t \in [0, 2)$$

$$v(t) = 0 \text{ when } t = 2$$

We now set $a(t) = 0$ on $[0, 2]$. We see that $a(t) = 0$ when $t^2 - 6t + 5 = 0$ which gives $t = 1, 5$. Since we are only looking at $t \in [0, 2]$, we only care about $t = 1$. Note that:

$$a(t) > 0 \text{ when } t \in [0, 1)$$

$$a(t) < 0 \text{ when } t \in (1, 2]$$

$$a(t) = 0 \text{ when } t = 1$$

The particle is speeding up when both $v(t)$ and $a(t)$ have the same sign. Thus, the particle is speeding up on the open interval $(0, 1)$.

(c) What is the maximum velocity of the particle in the time interval $0 \le t \le 2$?

Solution: To find the maximum velocity, we need to find when $v'(t) = a(t)$ is either equal to 0 or undefined on the interval $[0, 2]$. We already found that $a(t) = 0$ when $t = 1$ and that $a(t)$ is defined on all of $[0, 2]$. Thus, we have only one critical value, and that is $t = 1$. We also need to check the endpoints of the interval $t = 0, 2$. Evaluating $v(t)$ at these values gives:

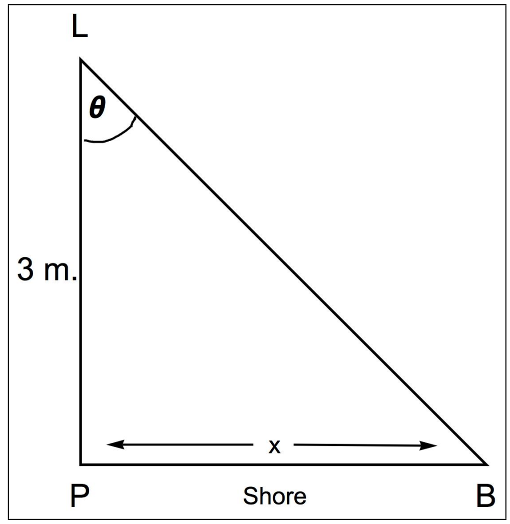
$$v(0) = 2/3, \ v(1) = 1, \ v(2) = 0$$

Thus, the maximum velocity is 1.

14.(12pts)

A lighthouse (L) is 3 miles from a point, P, on a straight shoreline (as shown in the diagram on the right). The light from the lighthouse revolves in an anti-clockwise direction at a speed of 6π radians per minute (that is three revolutions per minute).

How fast is the beam of light (B) moving along the shoreline when it is four miles from the point P?



Solutions: From the picture, we have

$$\tan\theta = \frac{x}{3},$$

where θ and x are functions of time t. Apply d/dt to the equation above yields

$$\sec^2(\theta)\theta' = \frac{x'}{3}.$$

When $x = 4$, $\tan\theta = \frac{4}{3}$, so $\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{16}{9} = \frac{25}{9}$.
On the other hand, from the description, we know that $\theta' = 6\pi$, hence,

$$\frac{25}{9}6\pi = \frac{x'}{3} \Rightarrow x' = 50\pi.$$

True-False.

15.(6pts) Please circle "TRUE" if you think the statement is true, and circle "FALSE" if you think the statement is False.

(a))(1 pt. No Partial credit) If $f(x)$ is a function which is differentiable at $x = a$, and $L(x)$ is the linearization of $f(x)$ at a, then $f(x) = L(x)$ for values of x near a. **Solution:** False. $f(x) \approx L(x)$ for values of x near a, but $L(x)$ need not be equal to $f(x)$ when x is near a.

TRUE FALSE

(b)(1 pt. No Partial credit) If f is a function and c is a number for which $f'(c) = 0$, then the graph of $y = f(x)$ must have a local maximum or a local minimum at $x = c$.

Solution: False. Consider $f(x) = x^3$ and $c = 0$. Note that $f'(c) = 0$, but $f(c)$ is neither a local max or min since $f(x) = x^3$ is an increasing function.

TRUE FALSE

(c))(1 pt. No Partial credit) If f is continuous on a closed interval [a, b], then it must attain an absolute maximum and minimum value on the interval $[a, b]$. Solution: True. This is the exact statement of the extreme value theorem.

TRUE FALSE

(d))(1 pt. No Partial credit) If f is a function for which $f'(x)$ exists and $f'(x)$ is not equal to zero for all x in the interval $(-\infty, +\infty)$, then $f(-1) \neq f(1)$.

Solution: True. If $f(-1) = f(1)$, then since f is differentiable, one could apply Rolle's Theorem/Mean Value Theorem to conclude that there would have to exist a value c such that $f'(c) = 0$. But the problem statement says that $f'(x) \neq 0$ for all x, thus a contradiction.

TRUE FALSE

(e))(2 pt. (1 pt. for answer, 1 pt. for justification)) A stone is thrown straight upwards with an initial height of 10 feet above the ground. The stone reaches the ground 5 seconds later. The stone must have had a velocity of -2 feet per second at some time during those 5 seconds.

Solutions: Apply Mean Value Theorem: $v(c) = \frac{s(5)-s(0)}{5-0} = \frac{0-10}{5} = -2$ for some $c \in (0, 5)$. Thus, the answer is True

TRUE FALSE

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15. Initials: ____________

16.(2pts) You will be awarded these two points if you write your name in CAPITALS on the front page and you mark your answers on the front page with an X through your answer choice like so: ~~(a)~~ (<u>not</u> an O around your answer choice) .

ROUGH WORK